Modeling Bond Prices In Continuous-Time
Part II - Solving For Risk-Free Bond Discount Rate

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In this white paper we will build a model that calculates the unknown market discount rate applicable to a risk-free bond with a known market value.

Our Hypothetical Problem

The table below presents our go-forward model assumptions from Part I...

Table 1: Bond Assumptions

Symbol	Description	Balance
P_0	Market price at time zero	\$958.82
B	Bond face value	\$1,000.00
C	Annual coupon rate (%)	4.50
T	Term in years (#)	3.00

We are tasked with answering the following questions:

Question 1: What is the continuous-time discount rate applicable to this risk-free bond?

Question 2: What is the yield to maturity and bond equivalent yield?

Bond Price Equations From Part I

In Part I we defined the variable P_0 to be the price at time zero of a coupon paying bond and the variable κ to be the continuous-time discount rate. Using Table 1 above the equation for bond price at time zero is... [1]

$$P_0 = \int\limits_0^T C\,B\,\mathrm{Exp}\left\{-\kappa\,u\right\}\delta u + B\,\mathrm{Exp}\left\{-\kappa\,T\right\} \tag{1}$$

The solution to Equation (1) above is... [1]

$$P_0 = B\left(C\,\kappa^{-1} - C\,\mathrm{Exp}\left\{-\kappa\,T\right\}\kappa^{-1} + \mathrm{Exp}\left\{-\kappa\,T\right\}\right) \tag{2}$$

The equation for the first derivative of bond price with respect to discount rate from Part I is... [1]

$$\frac{\delta}{\delta\kappa}\,P_0 = B\left(C\,\frac{\delta}{\delta\kappa}\,\kappa^{-1} - C\,\frac{\delta}{\delta\kappa}\,\mathrm{Exp}\left\{-\kappa\,T\right\}\kappa^{-1} + \frac{\delta}{\delta\kappa}\,\mathrm{Exp}\left\{-\kappa\,T\right\}\right) \tag{3}$$

The solution to Equation (3) above from Part I is... [1]

$$\frac{\delta}{\delta\kappa}\,P_0 = -B\left[C\left(1 + \mathrm{Exp}\left\{-\kappa\,T\right\}\left(1+\kappa\,T\right)\right)\kappa^{-2} + T\,\mathrm{Exp}\left\{-\kappa\,T\right\}\right] \tag{4}$$

Solving For The Discount Rate

We will define the variable r to be the actual discount rate (i.e. unknown to be solved for), the variable $\hat{r}$ to be a guess discount rate, the function $f(r)$ to be bond price at the actual discount rate (i.e. the observed bond price), the function $f(\hat{r})$ to be bond price at the guess discount rate, and the function $f'(\hat{r})$ to be the first derivative of bond price at the guess discount rate. Using these definitions we can solve for discount rate via the following Newton-Raphson method for solving nonlinear equations... [2]

$$\hat{r} + \frac{f(r) - f(\hat{r})}{f'(\hat{r})} = r + e \tag{5}$$

To solve for the actual discount rate we will come up with an initial guess rate and then iterate Equation (5) above until the error term e is zero (i.e. $r = \hat{r}$).

The Answer To Our Hypothetical Problem

Question 1: What is the continuous-time discount rate applicable to this risk-free bond?

Using Equations (2), (4) and (5) above the answer to our problem is...

Table 2: Newton-Raphson Solution

iteration	guess	f(guess)	f'(guess)	f(actual)		new guess
1	0.10000	857.45	-2388.69	958.82	=	0.05756
2	0.05756	965.39	-2704.83	958.82	=	0.05999
3	0.05999	958.84	-2685.64	958.82	=	0.06000
4	0.06000	958.82	-2685.58	958.82	=	0.06000
5	0.06000	958.82	-2685.58	958.82	=	0.06000

The discount rate used by the market to price this bond is 6.00%. We started with a guess rate of 10.00% and the solution took less than five iterations of the Newton-Raphson method to arrive at the actual rate of 6.00%.

Question 2: What is the yield to maturity and bond equivalent yield?

Using the answer to the question above the yield to maturity for this bond is...

$$\text{YTM} = \text{Exp}\left\{\kappa\right\} - 1 = \text{Exp}\left\{0.06000\right\} - 1 = 6.18\% \tag{6}$$

Using Equation (6) above the bond equivalent yield for this bond is...

$$\text{BEY} = 2 \times \left((1 + \text{YTM})^{0.5} - 1\right) = 2 \times \left((1 + 0.0618)^{0.5} - 1\right) = 6.09\% \tag{7}$$

Note: The bond pays coupon payments semi annually.

References

[1] Gary Schurman, *Modeling Bond Price in Continuous-Time - Part I,* November, 2020.

[2] Gary Schurman, *Newton-Raphson Method for Solving Nonlinear Equations - Part I,* October, 2009.